

# Quantization of Pseudoclassical Systems in the Schrödinger Realization

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We examine the quantization of pseudoclassical dynamical systems, models that have classically anticommuting variables, in the Schrödinger picture. We quantize these systems, which can be viewed as classical models of particle spin, using the Dirac-Gupta-Bleuler method as well as the reduced phase space method when applicable. We show that, with minimal modifications, the standard constructions of Schrödinger quantum mechanics for constrained systems work for pseudoclassical systems as well. In particular, we construct the space of spinors as physical wave functions of anticommuting variables.

## I. INTRODUCTION

Anticommuting variables, also called Grassmann numbers, have a long history in theoretical physics [1–5], with applications ranging from the path integral formulation of fermions to superspace constructions for supersymmetric theories. Pseudoclassical mechanics, which incorporates anticommuting dynamical variables, arises as the  $\hbar \rightarrow 0$  classical limit of quantum mechanical systems with spin [6, 7]. Despite the key role of anticommuting variables in theoretical physics, the Schrödinger picture approach for quantum mechanical systems described by anticommuting variables has received comparatively less attention.

In their renowned paper on the use of anticommuting variables to describe relativistic and non-relativistic spin degrees of freedom, Berezin and Marinov [7] posit a three dimensional vector-valued anticommuting variable  $\xi_k$  with the unlikely looking action

$$S = \int dt \left[ \frac{1}{2} \tilde{\omega}_{kl} \xi_k \dot{\xi}_l - H(\xi) \right], \quad (1.1)$$

with  $\tilde{\omega}$  an imaginary symmetric  $3 \times 3$  matrix, to describe the non-relativistic spin degrees of freedom of a spin-1/2 particle. Berezin and Marinov note from the form of the action that the variables  $\xi_k$  are evidently phase-space coordinates and then define a Poisson bracket that gives the correct equations of motion. After quantization, the operators  $\hat{\xi}_k$  corresponding to the pseudoclassical variables  $\xi_k$  become the generators of the Clifford algebra with three generators and satisfy the Pauli matrix anticommutation relations. Consistent with their abstract approach to mechanics, Berezin and Marinov appeal to the representation theory of Clifford algebras, and take the space of states to be the essentially unique irreducible

representation of that algebra, which is the space of two-component spinors.

While the abstract approach is certainly elegant, it sidesteps the usual methods of quantization and finesses the issues of dealing with actions that are first-order in time derivatives; it tells us what the answer is, not how to get there. Our purpose in this paper is to analyze pseudoclassical systems such as the ones introduced by Berezin and Marinov through the explicit application of Dirac's methods for dealing with constrained systems and the use of the Schrödinger representation for the states and their norms. This concrete realization of the more abstract approach offers insights into both pseudoclassical mechanics and constrained quantization, while also providing a methodology for analyzing pseudoclassical systems when representation theory might be insufficient.

## II. BACKGROUND

### A. Constrained Quantum Mechanics in the Schrödinger Representation

We begin by reviewing the standard approaches to quantization of constrained Hamiltonian systems [8–13], so that we establish clearly the machinery we will need when we turn to anticommuting variables.

To perform a Hamiltonian quantization of a classical theory defined through a Lagrangian  $L = L(q_i, \dot{q}_i)$  that depends on commuting coordinates  $q_i$  and their velocities  $\dot{q}_i$ , one first performs the Legendre transformation by inverting the definition of the canonical momenta,

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad (2.1)$$

to solve for the velocities

$$\dot{q}_i = \dot{q}_i(q, p), \quad (2.2)$$

and constructs the Hamiltonian,  $H(p, q) = \sum_i p_i \dot{q}_i - L(q, \dot{q})$ , where the velocities are understood as functions

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of positions and momenta. The positions and momenta are promoted to operators acting on states, which are square-integrable functions of position. The fundamental commutation relations between the position and momentum operators are determined by their Poisson brackets [14]

$$[\hat{q}_i, \hat{p}_j] = \hat{q}_i \hat{p}_j - \hat{p}_j \hat{q}_i = i\hbar \widehat{\{q_i, p_j\}}, \quad (2.3)$$

leading to the momenta being represented by derivatives with respect to coordinates. The inner product between two states is given by

$$\langle \phi | \psi \rangle = \int_{\mathbb{R}^N} \phi^*(q) \psi(q) dq. \quad (2.4)$$

The positions and momenta are observables. Observables must have real eigenvalues because classically observable quantities are real. One then checks that the positions and momenta are self-adjoint on the space of states with the inner product (2.4).

This story breaks down when the momenta (2.1) cannot be inverted to find all velocities in terms of positions and momenta. In such cases, there are one or more functional relations between the positions and momenta, called constraints,

$$\varphi_m(q, p) \approx 0, \quad (2.5)$$

which reduce the whole phase space from  $\mathbb{R}^{2N}$  to the submanifold  $\mathcal{C} = \{(q, p) \in \mathbb{R}^{2N} | \varphi_1(q, p) = \varphi_2(q, p) = \dots = \varphi_k(q, p) = 0\}$  that satisfies the relations (2.5). This submanifold is called the reduced phase space, or the “constraint surface.” The symbol  $\approx$  in (2.5) is read as “weakly zero,” meaning that it may only be set to zero after all derivatives have been taken. The Hamiltonian that guides the evolution of the system on  $\mathcal{C}$  may have several different functional forms in terms of the  $2N$  phase space variables, all of which agree numerically on the reduced phase space defined by (2.5),

$$H' = H + \lambda_m \varphi_m, \quad (2.6)$$

where a sum on the repeated index  $m$  is understood. One can fix the coefficients by requiring that the evolution of the system remain on the constraint surface, i.e.,

$$\frac{d\varphi_n}{dt} = \{\varphi_n, H'\} \approx 0. \quad (2.7)$$

There are two ways of quantizing a constrained classical system. In Dirac-Gupta-Bleuler quantization, one quantizes first, constructing the Hilbert space of square-integrable functions on the naive configuration space,  $L^2(\mathbb{R}^N)$ , and the position and momentum operators that satisfy Eq. (2.3), and then one constrains. The constraints (2.5) are promoted to operators

$$\hat{\varphi}_n = \varphi_n(\hat{q}, \hat{p}) \quad (2.8)$$

and used to define the physical Hilbert space,  $\mathfrak{H}_{\text{phys}} \subset L^2(\mathbb{R}^N)$ , on which the constraint operators (2.8) are the zero operator:

$$\langle \phi_{\text{phys}} | \hat{\varphi}_n | \psi_{\text{phys}} \rangle = 0 \quad \forall | \phi_{\text{phys}} \rangle, | \psi_{\text{phys}} \rangle \in \mathfrak{H}_{\text{phys}}. \quad (2.9)$$

In reduced phase space quantization, one constrains first, constructing the classical reduced phase space that satisfies the constraints and a generalized Poisson bracket—the Dirac bracket—of functions on that phase space. Then one quantizes by finding position and momentum coordinates on the reduced phase space, constructs the Hilbert space of square-integrable functions of the positions, and promotes the positions and momenta to operators satisfying the Dirac conditions (2.3).

When the matrix of Poisson brackets  $\{\varphi_n, \varphi_m\}$  is invertible,

$$\det(\{\varphi_n, \varphi_m\}) \neq 0, \quad (2.10)$$

as will be the case in systems we examine, the constraints (2.5) are called second-class. The reduced phase space  $\mathcal{C}$  is then even-dimensional,  $\dim \mathcal{C} = 2M$ , and a new bracket, the Dirac bracket, can be defined on functions on  $\mathcal{C}$ ,

$$\{f, g\}_{DB} = \{f, g\} - \{f, \varphi_n\} \Delta^{nm} \{\varphi_m, g\}, \quad (2.11)$$

where  $\Delta^{nm}$  is the inverse matrix to  $\{\varphi_n, \varphi_m\}$ . The constraints (2.5) can be taken to be strongly zero because the Dirac bracket of anything with a constraint vanishes identically,

$$\begin{aligned} \{f, \varphi_k\}_{DB} &= \{f, \varphi_k\} - \{f, \varphi_n\} \Delta^{nm} \{\varphi_m, \varphi_k\} \\ &= \{f, \varphi_k\} - \{f, \varphi_n\} \delta_k^n \equiv 0. \end{aligned} \quad (2.12)$$

The Dirac bracket has the same symmetry properties as the Poisson bracket and satisfies the Jacobi identity. We look for  $2M$  global coordinates  $z_I$  on  $\mathcal{C}$  such that  $M$  of them,  $Q_a$ , have zero Dirac brackets amongst themselves and so can be considered position coordinates. Each phase space coordinate is promoted to an operator satisfying (2.3), but with the Poisson brackets replaced by the Dirac ones. States are chosen to be square-integrable functions of the  $Q_a$ .

## B. Calculus of anticommuting variables

The calculus of anticommuting variables [4, 5, 15–17] is straightforward; we summarize the key points here. Functions of a finite number of anticommuting variables are defined through their power series; such a power series terminates, as anticommuting variables must be nilpotent (if  $\theta$  anticommutes with itself, then  $\theta\theta = -\theta\theta = 0$ ).

Derivatives must be specified as either being taken from the right or from the left. In either case, we have

$$\frac{\partial^R \theta}{\partial \theta} = \frac{\partial^L \theta}{\partial \theta} = 1. \quad (2.13)$$

Note that  $\partial^R/\partial\theta$  should be understood as sitting to the right of the function it is acting on. In order to preserve the “translation invariance” of the integral,

$$\int f(\theta) d\theta = \int f(\theta + \xi) d\theta, \quad (2.14)$$

we must have, up to a multiplicative constant,

$$\int d\theta = 0, \quad \int \theta d\theta = 1. \quad (2.15)$$

Both derivatives and measure factors anticommute with all anticommuting numbers and operators. For example, if  $\xi$  and  $\theta$  are anticommuting, we have

$$\begin{aligned} \frac{\partial^L}{\partial \theta}(\xi \theta) &= -\xi \frac{\partial^L}{\partial \theta} \theta = -\xi = -\frac{\partial^R}{\partial \theta}(\xi \theta), \quad \text{and} \\ \int \xi \theta d\xi &= -\int \xi d\xi \theta = -\theta. \end{aligned} \quad (2.16)$$

Complex conjugation, the classical correspondent of finding an adjoint, can be applied to expressions involving anticommuting variables. (Technically, this is an involution [16] of the Grassmann algebra, making it a  $*$ -algebra, but it generalizes complex conjugation in the commuting case.) It is usual, and an axiom of involutions of  $*$ -algebras, to take the complex conjugate of a product of two anticommuting variables to be

$$(\xi \theta)^* = \theta^* \xi^*, \quad (2.17)$$

which when combined with anticommutativity yields the unfamiliar result that the product of two real anticommuting numbers is imaginary. The properties of classical variables under complex conjugation carry over into the adjointness properties of their corresponding quantum operators.

### III. THE (TRIVIAL) CASE: ONE ANTICOMMUTING VARIABLE

We begin by considering the (trivial) case of a single real anticommuting variable, both to establish our methodology for the more interesting multivariable cases, and because it will appear embedded in some multivariable models. This case has also been examined by Bordi, Casalbuoni, and Barducci [18, 19].

With only one anticommuting variable, and absent anticommuting constant parameters, the only possible term in the Lagrangian is the kinetic term,

$$L = \frac{i}{2} \xi \dot{\xi}. \quad (3.1)$$

The equation of motion for  $\xi$  is that it is a constant. The momentum of the system does not depend on the velocity,

$$\pi = \frac{\partial^R L}{\partial \dot{\xi}} = \frac{i}{2} \xi, \quad (3.2)$$

so there is a constraint

$$\varphi = \pi - \frac{i}{2} \xi \approx 0, \quad (3.3)$$

and the only dynamics are that the system obeys the constraint, because the Hamiltonian vanishes identically. The phase space consists of the single variable  $\xi$ . Effectively, there is just a “half a degree of freedom.” The only way to quantize this system is to use the Dirac-Gupta-Bleuler quantization to impose the constraint. In the Schrödinger representation, the wave function is a linear function of  $\xi$ ,

$$\psi(\xi) = \psi_0 + \psi_1 \xi, \quad (3.4)$$

with  $\psi_0$  and  $\psi_1$  being complex numbers. Because the Poisson brackets, as we will see, are  $\{\pi, \xi\} = \{\xi, \pi\} = 1$ , the Dirac quantization rule gives the momentum operator (in  $\hbar = 1$  units)

$$\hat{\pi} = i \frac{\partial^L}{\partial \xi}. \quad (3.5)$$

We set the inner product to be the integral

$$\langle \phi | \psi \rangle = \int \phi^*(\xi) \psi(\xi) d\xi = \phi_1^* \psi_0 + \phi_0^* \psi_1. \quad (3.6)$$

Since the variable  $\xi$  is real,  $(\phi_0 + \phi_1 \xi)^* = \phi_0^* + \phi_1^* \xi$ . Dirac-Gupta-Bleuler quantization requires the constraint to have vanishing matrix elements between any two physical states

$$\langle \phi | \hat{\varphi} | \psi \rangle = i(\phi_1^* \psi_1 - \frac{1}{2} \phi_0^* \psi_0) = 0, \quad (3.7)$$

which implies that up to an overall phase, there is just a single normalized physical state of positive norm,

$$\psi_{\text{phys}}(\xi) = \frac{1}{\sqrt{2}} \left( 1 + \frac{1}{\sqrt{2}} \xi \right). \quad (3.8)$$

Note, finally, that we can convert this to a model with a single imaginary anticommuting variable by defining  $\xi' = i\xi$ . This changes the overall sign of the Lagrangian, and will lead to some factors of  $i$  in the inner product and the physical state and will change the relative sign in the constraint. This will play a role in Sections VI and VII. It is also worth mentioning that keeping the Lagrangian as (3.1) with a positive overall sign but positing an imaginary  $\xi$ , or having a negative sign for the Lagrangian and a real  $\xi$ , makes it impossible to impose the constraint (3.3) through the integral (3.7) because that expression becomes proportional to a positive definite expression,  $\langle \psi | \hat{\varphi} | \psi \rangle \propto \psi_1^* \psi_1 + \psi_0^* \psi_0 / 2$ .

## IV. TWO ANTICOMMUTING VARIABLES

### A. Two-variable Pseudoclassical System

We now consider the simplest non-trivial pseudoclassical system, one with two real Grassmann odd coordinates,  $\xi_1$  and  $\xi_2$ , and having the essentially unique Lagrangian

$$L = \frac{i}{2}(\xi_1 \dot{\xi}_1 + \xi_2 \dot{\xi}_2) + i\omega \xi_1 \xi_2. \quad (4.1)$$

The Euler-Lagrange equations of motion that follow from (4.1) are

$$\frac{d}{dt} \left( \frac{\partial^R L}{\partial \dot{\xi}_i} \right) = \frac{\partial^R L}{\partial \xi_i}, \quad (4.2)$$

or

$$\dot{\xi}_i = \omega(\xi_1 \delta_{i2} - \xi_2 \delta_{i1}) = -\omega \epsilon_{ij} \xi_j. \quad (4.3)$$

Passing to the Hamiltonian description requires finding the canonical momenta

$$\pi_i = \frac{\partial^R L}{\partial \dot{\xi}_i} = \frac{i}{2} \xi_i, \quad (4.4)$$

which lead immediately to constraints

$$\varphi_i = \pi_i - \frac{i}{2} \xi_i \approx 0, \quad (4.5)$$

which, because they do not have vanishing Poisson brackets with themselves, are second-class in Dirac's [8] classification.

The naive Hamiltonian is then

$$H = \pi_i \dot{\xi}_i - L = -i\omega \xi_1 \xi_2, \quad (4.6)$$

and the Poisson brackets are defined by

$$\{f, g\} = \sum_{i=1,2} \left( \frac{\partial^R f}{\partial \xi_i} \frac{\partial^L g}{\partial \pi_i} + \frac{\partial^R f}{\partial \pi_i} \frac{\partial^L g}{\partial \xi_i} \right). \quad (4.7)$$

The evolution of the system on the physical phase space defined by the constraints (4.5) should stay on that phase space, so the constraints must be conserved in time. The most general Hamiltonian that has the correct value (4.6) on the physical phase space is

$$H' = -i\omega \xi_1 \xi_2 + \lambda_i \varphi_i, \quad (4.8)$$

where the  $\lambda_i$  are Grassmann odd phase space functions.

The equations of motion arising from the Hamiltonian (4.8) are

$$\dot{\pi}_i = \{\pi_i, H\} = \frac{i}{2} \lambda_i - i\omega \epsilon_{ij} \xi_j + \frac{\partial^L \lambda_j}{\partial \xi_i} \varphi_j \approx \frac{i}{2} \lambda_i - i\omega \epsilon_{ij} \xi_j,$$

$$\dot{\xi}_i = \{\xi_i, H\} = -\lambda_i + \frac{\partial^L \lambda_j}{\partial \pi_i} \varphi_j \approx -\lambda_i. \quad (4.9)$$

The  $\lambda_i$  coefficients are determined by requiring that the constraints remain zero on the reduced phase space,

$$\dot{\varphi}_i = \{\varphi_i, H\} = i(\lambda_i - \omega \epsilon_{ij} \xi_j) + \left( \frac{\partial^L \lambda_j}{\partial \xi_i} - \frac{i}{2} \frac{\partial^L \lambda_j}{\partial \pi_i} \right) \varphi_j \approx 0. \quad (4.10)$$

It is enough for consistency to fix  $\lambda_i = \omega \epsilon_{ij} \xi_j$ , yielding the physical Hamiltonian

$$H_{\text{phys}} = \omega (\xi_2 \pi_1 - \xi_1 \pi_2) = -\omega \epsilon_{ij} \xi_i \pi_j, \quad (4.11)$$

which leads to the same equations of motion as the Euler-Lagrange equations of Eq. (4.3), at least on the constraint surface:

$$\dot{\xi}_i = \{\xi_i, H_{\text{phys}}\} = \frac{\partial^L H_{\text{phys}}}{\partial \pi_i} \approx -\omega \epsilon_{ij} \xi_j. \quad (4.12)$$

### B. Dirac-Gupta-Bleuler Quantization

#### 1. Operators and States

To quantize this system, we must replace the pseudoclassical phase space variables  $z_I$  (here,  $\xi_i$  and  $\pi_i$ ) by operators  $\hat{z}_I$  that satisfy the Dirac rule [14]

$$[\hat{z}_I, \hat{z}_J] = i\hbar \widehat{\{z_I, z_J\}}. \quad (4.13)$$

Here  $[\hat{z}_I, \hat{z}_J]$  is an anticommutator (despite the use of square brackets) because the phase space variables are Grassmann odd, and so the Poisson brackets are symmetric.

Applying the Dirac rule, we find

$$\begin{aligned} [\hat{\xi}_i, \hat{\xi}_j] &= 0, \\ [\hat{\xi}_i, \hat{\pi}_j] &= i\hbar \delta_{ij}, \\ [\hat{\pi}_i, \hat{\pi}_j] &= 0, \end{aligned} \quad (4.14)$$

which are satisfied by the operators

$$\begin{aligned} \hat{\xi}_i &= \xi_i, \\ \hat{\pi}_i &= i\hbar \frac{\partial^L}{\partial \xi_i}. \end{aligned} \quad (4.15)$$

In what follows, we again use units where  $\hbar = 1$ .

The wave functions for the states are functions of the coordinates  $\xi_i$ , which are defined by their power series,

$$\psi = \psi(\xi_1, \xi_2) = \psi_0 + \psi_1 \xi_1 + \psi_2 \xi_2 + \psi_3 \xi_1 \xi_2, \quad (4.16)$$

with complex coefficients  $\psi_i$ . Mathematically speaking, the wave functions take values in the Grassmann algebra over complex numbers.

## 2. Inner product and physical states

In addition to the standard axioms that an inner product on a Hilbert space must satisfy, namely  $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$  and linearity in the second argument,  $\langle\phi|\alpha\psi_1 + \beta\psi_2\rangle = \alpha\langle\phi|\psi_1\rangle + \beta\langle\phi|\psi_2\rangle$ , there are some further properties that an inner product on the space of physical quantum states must satisfy. These are:

1. the inner product must produce a positive norm for all physical states;
2. operators corresponding to observables must be self-adjoint under this inner product; and
3. if the system has constraints, then in the physical Hilbert space, the matrix elements of the second-class constraints must vanish under the inner product.

If the analogy between commuting and anticommuting coordinates held perfectly, the naive inner product on the space of all functions of the form (4.16) would be given by the integral over configuration space,

$$\int \phi^* \psi d\xi_1 d\xi_2 = \phi_3^* \psi_0 + \phi_2^* \psi_1 - \phi_1^* \psi_2 - \phi_0^* \psi_3, \quad (4.17)$$

but this would give  $\langle\phi|\psi\rangle = -\langle\psi|\phi\rangle^*$ , violating the axioms by giving states a manifestly imaginary norm. Heuristically speaking, the problem is that the ostensible “measure,”  $d\xi_1 d\xi_2$ , being a product of two real anticommuting expressions, is imaginary, a result of the rules of complex conjugation, (2.17). This is fixed by putting in an explicit factor of  $i$ , so that a satisfactory inner product is:

$$\langle\phi|\psi\rangle = i \int \phi^* \psi d\xi_1 d\xi_2. \quad (4.18)$$

Positive-definiteness, it is worth noting, does not need to hold on the full function space, but only on the space of physical states.

We use the third condition to identify the physical states. The constraint matrix elements between physical states must satisfy

$$\begin{aligned} \langle\phi|\hat{\varphi}_1|\psi\rangle &= -(\phi_3^* \psi_1 - \phi_1^* \psi_3) - \frac{1}{2}(\phi_0^* \psi_2 - \phi_2^* \psi_0) = 0, \\ \langle\phi|\hat{\varphi}_2|\psi\rangle &= -(\phi_3^* \psi_2 - \phi_2^* \psi_3) + \frac{1}{2}(\phi_0^* \psi_1 - \phi_1^* \psi_0) = 0. \end{aligned}$$

One might imagine that the way to satisfy these conditions would be to choose physical states to be those either of the form  $\psi_{\text{phys}} = \psi_1 \xi_1 + \psi_2 \xi_2$  or  $\psi_{\text{phys}} = \psi_0 + \psi_3 \xi_1 \xi_2$ ; in other words, physical states might be chosen to have just one definite Grassmann parity, since the matrix elements of a Grassmann odd operator between two states

of the same Grassmann parity vanish automatically. This choice is not correct.

It is not enough to have the matrix elements of the constraints vanish on the physical space of states; the physical states must have a probability interpretation and thus have positive norm. The norm of odd Grassmann parity states  $\psi_1 \xi_1 + \psi_2 \xi_2$  is  $i(\psi_2^* \psi_1 - \psi_1^* \psi_2)$ , which is positive under (4.18) when  $-i\psi_2/\psi_1$  is a positive real number. Similarly, the norm of even Grassmann parity states  $\psi_0 + \psi_3 \xi_1 \xi_2$  is  $i(\psi_3^* \psi_0 - \psi_0^* \psi_3)$ , which is positive under (4.18) when  $-i\psi_3/\psi_0$  is a positive real number.

The constraints are Grassmann odd, so the only matrix elements we need to compute are between Grassmann even and Grassmann odd states. We find that

$$\begin{aligned} \langle\phi_{\text{even}}|\hat{\varphi}_1|\psi_{\text{odd}}\rangle &= -(\phi_3^* \psi_1) - \frac{1}{2}(\phi_0^* \psi_2) = 0, \\ \langle\phi_{\text{even}}|\hat{\varphi}_2|\psi_{\text{odd}}\rangle &= -(\phi_3^* \psi_2) + \frac{1}{2}(\phi_0^* \psi_1) = 0 \end{aligned} \quad (4.19)$$

are satisfied when  $2(\phi_3/\phi_0)^* = -(\psi_2/\psi_1) = (\psi_1/\psi_2)$ .

By considering these conditions, we find an orthonormal basis for the wave functions of the full Schrödinger state space to be

$$\begin{aligned} |0\rangle &= 1 + \frac{i}{2}\xi_1 \xi_2, \\ |1\rangle &= \frac{1}{\sqrt{2}}(\xi_1 + i\xi_2), \\ |\bar{0}\rangle &= 1 - \frac{i}{2}\xi_1 \xi_2, \\ |\bar{1}\rangle &= \frac{1}{\sqrt{2}}(\xi_1 - i\xi_2). \end{aligned} \quad (4.20)$$

The inner product (4.18) gives

$$\langle 0|0\rangle = \langle 1|1\rangle = 1 = -\langle \bar{0}|\bar{0}\rangle = -\langle \bar{1}|\bar{1}\rangle, \quad (4.21)$$

and all other inner products vanish. Thus the full Schrödinger Hilbert space decomposes into a physical space and an orthogonal negative-norm “ghost” space,

$$\mathfrak{H}_{\text{Schrödinger}} = \mathfrak{H}_{\text{physical}} \oplus \mathfrak{H}_{\text{ghost}}. \quad (4.22)$$

The constraint operators each map the physical state space to the ghost state space and vice versa:

$$\begin{aligned} \hat{\varphi}_1|0\rangle &= -\frac{i}{\sqrt{2}}|\bar{1}\rangle, & \hat{\varphi}_1|1\rangle &= +\frac{i}{\sqrt{2}}|\bar{0}\rangle, \\ \hat{\varphi}_2|0\rangle &= +\frac{1}{\sqrt{2}}|\bar{1}\rangle, & \hat{\varphi}_2|1\rangle &= -\frac{1}{\sqrt{2}}|\bar{0}\rangle, \\ \hat{\varphi}_1|\bar{0}\rangle &= -\frac{i}{\sqrt{2}}|1\rangle, & \hat{\varphi}_1|\bar{1}\rangle &= +\frac{i}{\sqrt{2}}|0\rangle, \\ \hat{\varphi}_2|\bar{0}\rangle &= -\frac{1}{\sqrt{2}}|1\rangle, & \hat{\varphi}_2|\bar{1}\rangle &= +\frac{1}{\sqrt{2}}|0\rangle. \end{aligned} \quad (4.23)$$

The remaining condition on an inner product is that all observables be self-adjoint. Although anticommuting variables cannot be observables because they are nilpotent, it is nonetheless clear that  $\hat{\xi}_i$  is self-adjoint since

$$(\xi_i \psi(\xi_1, \xi_2))^* = \psi(\xi_1, \xi_2)^* \xi_i, \quad (4.24)$$

and we have  $\langle \phi | \hat{\xi}_i \psi \rangle = \langle \hat{\xi}_i \phi | \psi \rangle$ , or  $\hat{\xi}_i^\dagger = \hat{\xi}_i$ . Although  $\xi_i$  is real, its conjugate momentum  $\pi_i$  is not, as shown by the constraints  $\varphi_i = \pi_i - \frac{i}{2}\xi_i \approx 0$ . The momentum  $\hat{\pi}_i$  should therefore be anti-self-adjoint, which one can check by direct calculation:

$$\int \phi^* \left( i \frac{\partial^L}{\partial \xi_i} \psi \right) d\xi_1 d\xi_2 = - \int \left( i \frac{\partial^L}{\partial \xi_i} \phi \right)^* \psi d\xi_1 d\xi_2, \quad (4.25)$$

and so  $\hat{\pi}_i^\dagger = -\hat{\pi}_i$ .

The only observable in this system is the Hamiltonian corresponding to Eq. (4.11),

$$\hat{H}_{\text{phys}} = \omega(\hat{\xi}_2 \hat{\pi}_1 - \hat{\xi}_1 \hat{\pi}_2), \quad (4.26)$$

which we see is self-adjoint since  $(\hat{\xi}_2 \hat{\pi}_1)^\dagger = \hat{\pi}_1^\dagger \hat{\xi}_2^\dagger = -\hat{\pi}_1 \hat{\xi}_2 = +\hat{\xi}_2 \hat{\pi}_1$ , and similarly for the second term.

### 3. Energy spectrum

Because the Hamiltonian (4.26) comes from the Grassmann even Hamiltonian (4.11), the physical eigenstates can be taken to have definite Grassmann parity. We find

$$\begin{aligned} \hat{H}_{\text{phys}}|0\rangle &= 0, \\ \hat{H}_{\text{phys}}|1\rangle &= \omega|1\rangle. \end{aligned} \quad (4.27)$$

We remark that the ghost states, though unphysical, are also eigenstates of the Hamiltonian, with energies 0 and  $-\omega$ .

### 4. A strong Hamiltonian

In a generic constrained system, one cannot find a Hamiltonian of the form (4.11) for which the second-class constraints are identically conserved throughout the whole phase space, rather than just on the reduced phase space. Here, however, because the constraints  $\varphi_i$  are linear in the phase space variables, one might guess that choosing the coefficients  $\lambda_i$  to be linear in phase space variables could lead to a Hamiltonian that has Poisson brackets with either constraint that are strongly zero, not just weakly zero. We show that this is in fact possible in the case at hand, and that this method produces the same result.

Making the ansatz

$$\lambda_i = \beta_{ij}\xi_j + \gamma_{ij}\pi_j, \quad (4.28)$$

with  $\beta_{ij} = -\beta_{ji}$  and  $\gamma_{ij} = -\gamma_{ji}$ , we find

$$\lambda_i = \frac{3}{4}\omega\epsilon_{ij}\xi_j - \frac{i}{2}\omega\epsilon_{ij}\pi_j \quad (4.29)$$

and

$$H_{\text{phys}} = -\frac{i}{4}\omega\xi_1\xi_2 + \frac{1}{2}\omega(\xi_2\pi_1 - \xi_1\pi_2) + i\omega\pi_1\pi_2$$

$$= i\omega(\pi_1 + \frac{i}{2}\xi_1)(\pi_2 + \frac{i}{2}\xi_2). \quad (4.30)$$

This Hamiltonian has a shifted spectrum:

$$\begin{aligned} \hat{H}_{\text{phys}}|0\rangle &= -\frac{\omega}{2}|0\rangle, \\ \hat{H}_{\text{phys}}|1\rangle &= +\frac{\omega}{2}|1\rangle. \end{aligned} \quad (4.31)$$

Note that this is the same result as in the previous analysis (two states separated by energy  $\omega$ ), with a physically meaningless shift in the zero-point energy between these two approaches, which can be eliminated via some ordering conventions.

## C. Reduced Phase Space Quantization

### 1. Mechanics on the reduced phase space

Since the physical motion of a classical constrained system remains on the “constraint surface” where the second-class constraints vanish, it would be nice if one could set the constraints identically to zero both inside and outside of Poisson brackets and work purely with functions on the constraint surface, the reduced phase space. We execute this alternate approach to quantization of a constrained system here. To do this, the Poisson bracket on the unconstrained phase space must be replaced by the Dirac bracket on the reduced phase space, defined in Eq. (2.11).

In this case, because the matrix of Poisson brackets of the constraints is

$$\{\varphi_k, \varphi_\ell\} = -i\delta_{k\ell}, \quad (4.32)$$

the Dirac bracket becomes

$$\{f, g\}_{DB} = \{f, g\} - i\{f, \varphi_k\}\{\varphi_k, g\}. \quad (4.33)$$

The full phase space is four-dimensional while the constraint surface is two-dimensional. We could use just the two coordinates  $\xi_1$  and  $\xi_2$  as phase space coordinates on the constraint surface. Their Dirac brackets are

$$\begin{aligned} \{\xi_i, \xi_j\}_{DB} &= \{\xi_i, \xi_j\} - i\{\xi_i, \varphi_k\}\{\varphi_k, \xi_j\} \\ &= 0 - i\delta_{ik}\delta_{kj} = -i\delta_{ij}, \end{aligned} \quad (4.34)$$

so that the Dirac bracket of functions  $f(\xi_1, \xi_2)$  and  $g(\xi_1, \xi_2)$  on the constraint surface is

$$\{f, g\}_{DB} = -i \sum_{k=1,2} \left( \frac{\partial^R f}{\partial \xi_k} \frac{\partial^L g}{\partial \xi_k} \right). \quad (4.35)$$

We note that up to an overall sign, the Dirac bracket Eq. (4.35) is the abstract Poisson bracket postulated by Berezin and Marinov, at least for two anticommuting variables.

## 2. Operators and states

As  $\xi_1$  and  $\xi_2$  are coordinates of the two-dimensional reduced phase space, for quantization in the Schrödinger picture one must choose one position coordinate and one canonical momentum to proceed. Neither  $\xi_1$  nor  $\xi_2$  can fulfill either role; each has non-vanishing Dirac bracket with itself.

Instead, consider the complex phase space coordinates,

$$\begin{aligned}\eta &= \frac{1}{\sqrt{2}}(\xi_1 + i\xi_2), \\ \bar{\eta} &= \frac{1}{\sqrt{2}}(\xi_1 - i\xi_2),\end{aligned}\quad (4.36)$$

which satisfy

$$\{\eta, \eta\}_{DB} = \{\bar{\eta}, \bar{\eta}\}_{DB} = 0, \quad \{\eta, \bar{\eta}\}_{DB} = -i. \quad (4.37)$$

Generalizing the Dirac rule (4.13), we need operators that satisfy

$$[\hat{\eta}, \hat{\eta}] = \hat{\eta}\hat{\eta} + \hat{\eta}\hat{\eta} = i\hbar\widehat{\{\eta, \eta\}_{DB}} = \hbar. \quad (4.38)$$

We can proceed with quantization in the Schrödinger picture if we take states to be functions of  $\eta$  alone,

$$\psi = \psi(\eta) = \psi_0 + \psi_1\eta, \quad (4.39)$$

and the operators  $\hat{\eta}$  and  $\hat{\bar{\eta}}$  acting upon them to be

$$\begin{aligned}\hat{\eta} &= \eta, \\ \hat{\bar{\eta}} &= \hbar \frac{\partial^L}{\partial \eta}.\end{aligned}\quad (4.40)$$

Again, we use units in which  $\hbar = 1$  in what follows. As we are working in the reduced phase space, the constraints were eliminated before quantization, so all we need do now is construct the inner product and find the spectrum.

## 3. Inner product

We might like to mimic the standard inner product, but while the wave function is a function of  $\eta$ , its complex conjugate is a function of  $\bar{\eta}$ . Thus we are forced to consider inner products of the form

$$\langle \phi | \psi \rangle = \int \phi^*(\bar{\eta}) \psi(\eta) \mathcal{M}(\bar{\eta}, \eta) d\eta d\bar{\eta}, \quad (4.41)$$

where  $\mathcal{M}(\bar{\eta}, \eta)$  is a measure factor needed to enforce the adjointness relations coming from the complex conjugate nature of the variables  $\bar{\eta}$  and  $\eta$ ,  $\eta^* = \bar{\eta}$ . We need to have

$$\hat{\eta}^\dagger = \hat{\bar{\eta}} = \frac{\partial^L}{\partial \eta}, \quad (4.42)$$

or, for any two states  $\phi$  and  $\psi$ ,

$$\begin{aligned}\langle \hat{\eta} \phi | \psi \rangle &= \int \left( \frac{\partial^L \phi}{\partial \eta} \right)^* \psi(\eta) \mathcal{M}(\bar{\eta}, \eta) d\eta d\bar{\eta} \\ &= \int \phi^*(\bar{\eta}) \eta \psi(\eta) \mathcal{M}(\bar{\eta}, \eta) d\eta d\bar{\eta} \\ &= \langle \phi | \hat{\eta} \psi \rangle.\end{aligned}\quad (4.43)$$

Similarly, we need  $\hat{\eta} = \hat{\bar{\eta}}^\dagger$ , or

$$\begin{aligned}\langle \hat{\eta} \phi | \psi \rangle &= \int (\eta \phi)^* \psi(\eta) \mathcal{M}(\bar{\eta}, \eta) d\eta d\bar{\eta} \\ &= \int \phi^*(\bar{\eta}) \bar{\eta} \psi(\eta) \mathcal{M}(\bar{\eta}, \eta) d\eta d\bar{\eta} \\ &= \int \phi^*(\bar{\eta}) \frac{\partial^L \psi}{\partial \eta} \mathcal{M}(\bar{\eta}, \eta) d\eta d\bar{\eta} \\ &= \langle \phi | \hat{\bar{\eta}} \psi \rangle.\end{aligned}\quad (4.44)$$

Putting the general  $\mathcal{M}(\bar{\eta}, \eta) = \mathcal{M}_{00} + \mathcal{M}_{10}\bar{\eta} + \mathcal{M}_{01}\eta + \mathcal{M}_{11}\bar{\eta}\eta$  into the conditions, we find that the adjointness conditions, Eqs. (4.43) and (4.44), are identities if and only if  $\mathcal{M}_{00} = \mathcal{M}_{11}$ , and  $\mathcal{M}_{01} = \mathcal{M}_{10} = 0$ . Up to an overall factor, then, we have

$$\mathcal{M}(\bar{\eta}, \eta) = 1 + \bar{\eta}\eta = \exp(\bar{\eta}\eta), \quad (4.45)$$

and the inner product on states  $\psi(\eta) = \psi_0 + \psi_1\eta$  and  $\phi(\eta) = \phi_0 + \phi_1\eta$  is

$$\langle \phi | \psi \rangle = \int \phi^* \psi \exp(\bar{\eta}\eta) d\eta d\bar{\eta} = \psi_0^* \phi_0 + \psi_1^* \phi_1. \quad (4.46)$$

This inner product leads to positive definite norms for states. Because the constraints have been implemented prior to quantization, there is no ghost sector.

## 4. Energy spectrum

The Hamiltonian,  $H = -i\omega\xi_1\xi_2 = \omega\eta\bar{\eta}$  becomes

$$\hat{H} = \omega\hat{\eta}\hat{\bar{\eta}} = \omega\eta\frac{\partial^L}{\partial \eta}. \quad (4.47)$$

The eigenstates and spectrum are

$$\begin{aligned}|0\rangle &= 1, \\ |1\rangle &= \eta, \\ \hat{H}|0\rangle &= 0, \\ \hat{H}|1\rangle &= \omega|1\rangle,\end{aligned}\quad (4.48)$$

in agreement with the spectrum (4.27) found for the Dirac-Gupta-Bleuler quantization. Unlike the Hamiltonian (4.30), the reduced phase space Hamiltonian (4.47) has an ordering ambiguity. If (4.47) is Weyl ordered,

$$\hat{H}_{\text{Weyl}} = \omega \frac{1}{2} (\hat{\eta}\hat{\bar{\eta}} + \hat{\bar{\eta}}\hat{\eta}) = \frac{\omega}{2} \left( \eta \frac{\partial^L}{\partial \eta} + \frac{\partial^L}{\partial \eta} \eta \right), \quad (4.49)$$

then the spectrum is symmetric about zero, like that in Eq. (4.31).

We note that the basis eigenstates in this system,  $|0\rangle$  and  $|1\rangle$ , have wave functions with definite Grassmann parity, which correspond to the Grassmann parities of the equivalent states found under Dirac-Gupta-Bleuler quantization. The similarity between the states of the two different quantizations is stronger than just their Grassmann parities, however.

#### D. Dirac-Gupta-Bleuler and Reduced Phase Space Wave Function Correspondence

If the configuration space, rather than the reduced phase space, is parametrized by the  $\eta$  and  $\bar{\eta}$  coordinates of Eq. (4.36), we may rewrite the physical Dirac-Gupta-Bleuler wave functions given in Eq. (4.20) as the reduced phase space ones times the square root of the reduced phase space measure factor,

$$1 + \frac{i}{2}\xi_1\xi_2 = 1 + \frac{1}{2}\bar{\eta}\eta = \sqrt{e^{\bar{\eta}\eta}} = \sqrt{\mathcal{M}_{\text{RPS}}},$$

$$\frac{1}{\sqrt{2}}(\xi_1 + i\xi_2) = \eta = \eta\sqrt{e^{\bar{\eta}\eta}} = \eta\sqrt{\mathcal{M}_{\text{RPS}}}, \quad (4.50)$$

which is to say,

$$(\psi_n(\xi_1, \xi_2))_{\text{DGB}} = (\psi_n(\eta)\sqrt{\mathcal{M}})_{\text{RPS}}. \quad (4.51)$$

The inner product on the physical Dirac-Gupta-Bleuler space of states is the integral over the  $\xi_1, \xi_2$  configuration space, which can be reparametrized as an integral over the  $\eta, \bar{\eta}$  configuration space, making the orthonormality of the one set understandable in terms of the other.

### V. THREE ANTICOMMUTING VARIABLES

#### A. Three-variable Pseudoclassical System

We now generalize to the case of three anticommuting variables—the first case treated by Berezin and Marinov [7]. After diagonalization of the kinetic terms, the most general Lagrangian is

$$L = \frac{i}{2}\xi_k\dot{\xi}_k + i\omega_k\epsilon_{ijk}\xi_i\xi_j, \quad (5.1)$$

which contains three arbitrary commuting constants,  $\omega_k$ . A further rotation of the  $\xi_i$  and  $\omega_k$  allows the reduction of the Lagrangian to

$$L = \frac{i}{2}\xi_k\dot{\xi}_k + i\omega\xi_1\xi_2, \quad (5.2)$$

which has the same form as the Lagrangian (4.1), except now the kinetic term contains the additional piece  $\frac{i}{2}\xi_3\dot{\xi}_3$ .

As a consequence, we might try to anticipate the result of the explicit quantization. Since the Lagrangian (5.2) separates into two non-interacting parts, one involving  $\xi_1$  and  $\xi_2$  and having the form of the two-variable system analyzed in the preceding section, and the other involving  $\xi_3$  and having the form analyzed in Section IV, the basis states of the three-variable system can be written in terms of products of the basis states of those two simpler systems. The Hamiltonian that commutes with the constraints will be identical to (4.30).

In particular, we might expect that the basis states for the physical sector of the three-variable system can be obtained by taking a product of two physical basis states or two ghost basis states, one from the two-variable system and one from the one-variable system. Likewise, the ghost basis states for the three-variable system would be written as the product of a physical basis state and a ghost basis state, one from the two-variable system and one from the one-variable system. It turns out that this is a correct description of what happens with the Dirac-Gupta-Bleuler quantization, but there is a reduced phase space approach that gives a slightly different result, as we will see.

Note that when we compare the three-variable system to the two-variable system, two of the constraints and two of the equations of motion are the same but there is one additional constraint, which has the same form as the other two constraints,

$$\varphi_3 = \pi_3 - \frac{i}{2}\xi_3 \approx 0, \quad (5.3)$$

and one additional equation of motion,

$$\dot{\xi}_3 = 0. \quad (5.4)$$

As we know, the one-variable system has a Hamiltonian that vanishes identically, and so the Hamiltonian for the three-variable system has the same form as Eq. (4.30), although the wave functions can have  $\xi_3$  dependence.

We now give the results of explicit quantization.

#### B. Dirac-Gupta-Bleuler quantization

##### 1. States

With three Grassmann coordinates, the “measure” will now be the product  $i d\xi_1 d\xi_2 d\xi_3$ , which is Grassmann odd. This means that the normalizable states cannot be taken to have a definite Grassmann parity. For the system described by (5.2), the wave functions of the system can be factorized as

$$\Psi(\xi_1, \xi_2, \xi_3) = \psi(\xi_1, \xi_2) u(\xi_3). \quad (5.5)$$

If the two-dimensional functions  $\psi(\xi_1, \xi_2)$  have definite Grassmann parity, then it is easy to see that the matrix elements of the first two second-class constraints will vanish if the two-dimensional factors of the wave functions are either both in  $\mathfrak{H}_{\text{physical}}$  or both in  $\mathfrak{H}_{\text{ghost}}$  of the two-variable Hilbert space (4.22);

$$\begin{aligned}\langle \Phi | \hat{\varphi}_{1,2} | \Psi \rangle &= i \int (\phi v)^* \hat{\varphi}_{1,2} \psi u d\xi_1 d\xi_2 d\xi_3 \\ &= i \int (v^* \tilde{u}) (\phi^* \hat{\varphi}_{1,2} \psi) d\xi_1 d\xi_2 d\xi_3 \\ &= 0,\end{aligned}\quad (5.6)$$

where  $\tilde{u}$  is either  $u(-\xi_3)$  or  $u(\xi_3)$ , depending on whether the Grassmann parities of  $\phi(\xi_1, \xi_2)$  and  $\psi(\xi_1, \xi_2)$  are the same or different respectively. The matrix elements of the third constraint are

$$\begin{aligned}\langle \Phi | \hat{\varphi}_3 | \Psi \rangle &= i \int (\phi v)^* \hat{\varphi}_3 \psi u d\xi_1 d\xi_2 d\xi_3 \\ &= i \int v^* \phi^* \hat{\varphi}_3 \psi u d\xi_1 d\xi_2 d\xi_3 \\ &= i \int (v^* \hat{\varphi}_3 \tilde{u}) (\phi^* \psi) d\xi_1 d\xi_2 d\xi_3 \\ &= i \int (v^* \hat{\varphi}_3 \tilde{u}) d\xi_3 \int \phi^* \psi d\xi_1 d\xi_2,\end{aligned}\quad (5.7)$$

where  $\tilde{u}(\xi_3)$  is  $(-1)^{g_\phi} u((-1)^{g_\phi + g_\psi} \xi_3)$ , where  $g_\phi$  and  $g_\psi$  denote the Grassmann parities of  $\phi$  and  $\psi$ , respectively. The second factor,  $\int \phi^* \psi d\xi_1 d\xi_2$ , vanishes unless  $g_\phi = g_\psi$ . In that case,  $\tilde{u} = (-1)^{g_\phi} u$  and there are two solutions that make the matrix elements (5.7) vanish, both of which have  $u = v$ , namely

$$u(\xi_3) = v(\xi_3) = \frac{1}{\sqrt[4]{2}} \left( 1 \pm \frac{\xi_3}{\sqrt{2}} \right). \quad (5.8)$$

The norm of a product wave function  $\psi(\xi_1, \xi_2) u(\xi_3)$  is the product of the norms of its factors,

$$\begin{aligned}\langle \psi u | \psi u \rangle &= i \int u^* \psi^* \psi u d\xi_1 d\xi_2 d\xi_3 \\ &= \left( \int u^* u d\xi_3 \right) \left( i \int \psi^* \psi d\xi_1 d\xi_2 \right).\end{aligned}\quad (5.9)$$

Consequently, the positive norm physical states are spanned by the orthonormal basis

$$\begin{aligned}|0\rangle &= \frac{1}{\sqrt[4]{2}} \left( 1 + \frac{i}{2} \xi_1 \xi_2 \right) \left( 1 + \frac{\xi_3}{\sqrt{2}} \right), \\ |1\rangle &= \frac{1}{\sqrt[4]{8}} (\xi_1 + i \xi_2) \left( 1 + \frac{\xi_3}{\sqrt{2}} \right), \\ |0'\rangle &= \frac{1}{\sqrt[4]{2}} \left( 1 - \frac{i}{2} \xi_1 \xi_2 \right) \left( 1 - \frac{\xi_3}{\sqrt{2}} \right), \\ |1'\rangle &= \frac{1}{\sqrt[4]{8}} (\xi_1 - i \xi_2) \left( 1 - \frac{\xi_3}{\sqrt{2}} \right).\end{aligned}\quad (5.10)$$

The negative norm ghost states are spanned by the orthogonal anti-normal basis

$$|\bar{0}\rangle = \frac{1}{\sqrt[4]{2}} \left( 1 - \frac{i}{2} \xi_1 \xi_2 \right) \left( 1 + \frac{\xi_3}{\sqrt{2}} \right),$$

$$\begin{aligned}|\bar{1}\rangle &= \frac{1}{\sqrt[4]{8}} (\xi_1 - i \xi_2) \left( 1 + \frac{\xi_3}{\sqrt{2}} \right), \\ |\bar{0}'\rangle &= \frac{1}{\sqrt[4]{2}} \left( 1 + \frac{i}{2} \xi_1 \xi_2 \right) \left( 1 - \frac{\xi_3}{\sqrt{2}} \right), \\ |\bar{1}'\rangle &= \frac{1}{\sqrt[4]{8}} (\xi_1 + i \xi_2) \left( 1 - \frac{\xi_3}{\sqrt{2}} \right).\end{aligned}\quad (5.11)$$

We see that the large Schrödinger Hilbert space again splits as in Eq. (4.22), but this time both the physical and ghost spaces have dimension four. The physical Hilbert space forms the reducible  $\mathbf{2} \oplus \mathbf{2}$  representation of the three-dimensional Clifford algebra. By contrast, the ghost Hilbert space forms the reducible representation  $\bar{\mathbf{2}} \oplus \bar{\mathbf{2}}$  of the three-dimensional Clifford algebra. We will see that the  $\mathbf{2}$  and the  $\bar{\mathbf{2}}$  are irreducible representations of the Pauli matrix algebra,

$$\sigma_j \sigma_k = \mp i \epsilon_{jkl} \sigma_l, \quad (5.12)$$

for a left-handed and a right-handed coordinate system respectively.

## 2. Physical spectrum

The states (5.10) are eigenstates of the Hamiltonian:

$$\begin{aligned}\hat{H}_{\text{phys}} |0\rangle &= -\frac{\omega}{2} |0\rangle, \\ \hat{H}_{\text{phys}} |1\rangle &= +\frac{\omega}{2} |1\rangle, \\ \hat{H}_{\text{phys}} |0'\rangle &= -\frac{\omega}{2} |0'\rangle, \\ \hat{H}_{\text{phys}} |1'\rangle &= +\frac{\omega}{2} |1'\rangle.\end{aligned}\quad (5.13)$$

## 3. Matrix elements of $\hat{\xi}_i$ and the Pauli matrices

The integrals for the matrix elements of the position operators  $\hat{\xi}_i$  can be worked out. We find the interesting result that

$$\begin{aligned}\begin{pmatrix} \langle 0 | \hat{\xi}_1 | 0 \rangle & \langle 0 | \hat{\xi}_1 | 1 \rangle \\ \langle 1 | \hat{\xi}_1 | 0 \rangle & \langle 1 | \hat{\xi}_1 | 1 \rangle \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \begin{pmatrix} \langle 0 | \hat{\xi}_2 | 0 \rangle & \langle 0 | \hat{\xi}_2 | 1 \rangle \\ \langle 1 | \hat{\xi}_2 | 0 \rangle & \langle 1 | \hat{\xi}_2 | 1 \rangle \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \\ \begin{pmatrix} \langle 0 | \hat{\xi}_3 | 0 \rangle & \langle 0 | \hat{\xi}_3 | 1 \rangle \\ \langle 1 | \hat{\xi}_3 | 0 \rangle & \langle 1 | \hat{\xi}_3 | 1 \rangle \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\end{aligned}\quad (5.14)$$

It is instructive to note that the diagonal entries in the matrix in the last equation of (5.14) result from the even or odd definite Grassmann parities of the  $\psi(\xi_1, \xi_2)$  pieces (5.5) of the basis states  $|0\rangle$  and  $|1\rangle$  of (5.10). These matrix elements are the left-handed Pauli matrices; the matrix elements in a ghost basis are the usual (right-handed) Pauli matrices. The two representations are inequivalent because the operator representing  $-i\sigma_1\sigma_2\sigma_3$ ,

$-i2\sqrt{2}\hat{\xi}'_1\hat{\xi}'_2\hat{\xi}'_3$ , is  $-1$  on the physical states and  $+1$  on the ghost states. While the matrix elements yield Pauli matrices, the  $\hat{\xi}'_k$  themselves do not form a Clifford algebra; they are still nilpotent generators of a Grassmann algebra. However, the  $\hat{\xi}'_k$  operators

$$\hat{\xi}'_k = -i\hat{\pi}_k + \frac{1}{2}\hat{\xi}_k = \frac{\partial^L}{\partial \xi_k} + \frac{1}{2}\xi_k, \quad (5.15)$$

do form a Clifford algebra and correspond to the (scaled) Pauli matrices  $\frac{1}{\sqrt{2}}\sigma_k$ . The  $\hat{\xi}'_k$  are the quantized Dirac primed [9] quantities

$$\xi'_k = \xi_k - \{\xi_k, \varphi_i\} \Delta^{ij} \varphi_j, \quad (5.16)$$

where  $\Delta^{ij}$  is the inverse matrix to  $\{\varphi_i, \varphi_j\}$ . Dirac primed quantities have Poisson brackets that are at least weakly equal to the Dirac brackets of the unprimed quantities. In this case the  $\xi'_k$  have Poisson brackets strongly equal to the Dirac brackets of the  $\xi_k$ . These primed variables also show up in the Hamiltonian (4.30).

### C. Hybrid reduced phase space quantization

Because the reduced phase space is three-dimensional, it is impossible to perform a genuine reduced phase space quantization. But because of the relationship of this model to the two- and one-variable systems, it is possible here to bootstrap from the reduced phase space quantization with two variables and then append the unique normalizable physical state (3.8) for the variable  $\xi_3$ , yielding

$$\Psi(\eta, \xi_3) = \psi(\eta) u(\xi_3), \quad (5.17)$$

which results in a physical state. Denoting  $\tilde{\phi}^*(\bar{\eta}) = \phi^*(-\bar{\eta})$ , we find that

$$\begin{aligned} \langle \Phi | \hat{\varphi}_3 | \Psi \rangle &= \int v^*(\xi_3) \phi^*(\bar{\eta}) \hat{\varphi}_3 \psi(\eta) u(\xi_3) e^{\bar{\eta}\eta} d\eta d\bar{\eta} d\xi_3 \\ &= \int (v^* \hat{\varphi}_3 u) (\tilde{\phi}^* \psi e^{\bar{\eta}\eta}) d\eta d\bar{\eta} d\xi_3 \\ &= 0, \end{aligned} \quad (5.18)$$

when both  $u$  and  $v$  are the wave functions given in Eq. (5.8). Note that the second equality in Eq. (5.18) holds for the integral but not for the integrands themselves.

The Hamiltonian is identical to (4.47) and the (unnormalized) eigenstates of the Hamiltonian and their energies are

$$\begin{aligned} |0\rangle &= (1 + \frac{1}{\sqrt{2}}\xi_3), \\ |1\rangle &= \eta(1 + \frac{1}{\sqrt{2}}\xi_3), \\ \hat{H}|0\rangle &= 0, \end{aligned}$$

$$\hat{H}|1\rangle = \omega|1\rangle. \quad (5.19)$$

The hybrid reduced phase space quantization produces a physical Hilbert space that is an irreducible representation of the Clifford algebra with three generators. A ghost sector arises if we append the unique ghost state for the third variable  $\xi_3$  to the physical states. The irreducible representations of these physical and ghost sectors are those of the left-handed and right-handed Pauli matrices, respectively. (If we had chosen  $\bar{\eta}$  to be the position and  $\eta$  its conjugate momentum, then the physical states and ghost states would be irreducible representations of the right-handed and left-handed Pauli matrices, respectively.)

While this method produces a physical Hilbert space consisting of a single irreducible representation, as we saw previously, the Dirac-Gupta-Bleuler quantization produces physical Hilbert space consisting of two independent copies of this irreducible representation (with a similar doubling in the ghost sector, too). In the Dirac-Gupta-Bleuler approach, the two copies of the irreducible representation constitute distinct superselection sectors — the quantum operators do not map between them — so it is logically allowable to ignore one of the sectors (which would produce agreement with the reduced phase space quantization), but it is not necessary to do so. That the two methods of quantization can yield different results is not a problem mathematically, as although it is well known that while they often do agree, they are not guaranteed to do so [20–22]. In fact, it can happen [23] that the energy spectra differ more radically than just by a degeneracy, so the situation here is relatively tame. While both quantizations may be mathematically consistent, they are physically different in such cases, and one would need to invoke additional criteria to prefer one over the other.

## VI. LORENTZIAN METRIC

We now ask what happens to the system if the metric for the  $\xi$  variables is not Euclidean but Lorentzian. In other words, what happens if one of the kinetic terms changes sign? In the simplest such case, we have two coordinates and as before, up to an overall constant, the most general Lagrangian we can write is

$$\begin{aligned} L &= \frac{i}{2}(\xi_3 \dot{\xi}_3 - \xi_0 \dot{\xi}_0) - iB\xi_0\xi_3 \\ &= \frac{i}{2}\xi_\mu \dot{\xi}^\mu - iB\xi_0\xi_3, \end{aligned} \quad (6.1)$$

where  $B$  is a constant commuting number.

The Lagrangian (6.1) describes a physically dubious system; the coordinates  $\xi_\mu$ , instead of undergoing rotation with a constant angular velocity as they do under

(4.1), undergo a boost with constant rate of change,  $B$ , of rapidity. We will see below that the energy of this system is apparently not real and if the rate of change of rapidity is made imaginary to make the energies real, then it is not clear what is represented by this system. Nevertheless, it is instructive to consider this case because a theory with a Lorentzian kinetic energy can be used to describe the important case of a pseudoclassical version of the Dirac equation [7, 24–26].

The equations of motion that follow from the Lagrangian (6.1) are

$$\dot{\xi}_\mu = -B(\delta_\mu^3 \xi_0 + \delta_\mu^0 \xi_3). \quad (6.2)$$

The Lagrangian (6.1) leads as before to the second class constraints

$$\varphi_\mu = \pi_\mu - \frac{i}{2} \xi_\mu \approx 0, \quad (6.3)$$

and the general Hamiltonian

$$H' = iB\xi_0\xi_1 + \lambda^\mu \varphi_\mu. \quad (6.4)$$

The evolution of the system stays on the constraint surface if the Poisson bracket of each constraint with the Hamiltonian gives zero,

$$\begin{aligned} \dot{\varphi}_\mu &= \{\varphi_\mu, H'\} \\ &= i\lambda_\mu + iB(\delta_\mu^3 \xi_0 - \delta_\mu^0 \xi_3) + \left( \frac{\partial^L \lambda^\nu}{\partial \xi^\mu} - \frac{i}{2} \frac{\partial^L \lambda^\nu}{\partial \pi^\mu} \right) \varphi_\nu \\ &= 0. \end{aligned} \quad (6.5)$$

Because Eqs. (6.5) can be made linear in the  $\xi$  and  $\pi$  variables if the  $\lambda^\nu$  are as well, we can again make an ansatz of the form (4.28), and solve (6.5) for the  $\lambda^\nu$ , which leads to the Hamiltonian

$$\begin{aligned} H' &= \frac{i}{4} B \xi_0 \xi_3 + \frac{1}{2} B (\xi_0 \pi_3 - \xi_3 \pi_0) - iB \pi_0 \pi_3 \\ &= -iB(\pi_0 + \frac{i}{2} \xi_0)(\pi_3 + \frac{i}{2} \xi_3), \end{aligned} \quad (6.6)$$

which strongly conserves all constraints.

Using the Poisson brackets in this case,

$$\{f, g\} = \frac{\partial^R f}{\partial \xi_\mu} \frac{\partial^L g}{\partial \pi^\mu} + \frac{\partial^R f}{\partial \pi^\mu} \frac{\partial^L g}{\partial \xi_\mu}, \quad (6.7)$$

we find that the Hamiltonian equations of motion,

$$\dot{\xi}_\mu = \{\xi_\mu, H'\} = \frac{\partial^L H'}{\partial \pi^\mu} = \eta_{\mu\nu} \frac{\partial^L H'}{\partial \pi_\nu} \quad (6.8)$$

yield

$$\begin{aligned} \dot{\xi}_\mu &= B \left( \delta_\mu^0 \left( i\pi_3 - \frac{1}{2} \xi_3 \right) + \delta_\mu^3 \left( i\pi_0 - \frac{1}{2} \xi_0 \right) \right) \\ &\approx -B(\delta_\mu^0 \xi_3 + \delta_\mu^3 \xi_0), \end{aligned} \quad (6.9)$$

which can be taken to be strong equations as the Hamiltonian  $H'$  conserves all the constraints.

## A. Dirac-Gupta-Bleuler Quantization

As before, we turn  $\pi$  and  $\xi$  into operators satisfying the Dirac rule (4.13), with

$$\begin{aligned} \hat{\xi}_\mu &= \xi_\mu, \\ \hat{\pi}_\mu &= i \frac{\partial^L}{\partial \xi^\mu} = i \eta_{\mu\nu} \frac{\partial^L}{\partial \xi_\nu}. \end{aligned} \quad (6.10)$$

The general wave function can be written as

$$\psi = \psi_0 + \psi^\mu \xi_\mu + \psi^{03} \xi_0 \xi_3. \quad (6.11)$$

Considering the time-independent Schrödinger equation  $\hat{H}'\psi = E\psi$  on Grassmann even states, we find

$$\begin{aligned} \hat{H}'\psi &= -iB \left( \frac{\partial}{\partial \xi_0} - \frac{1}{2} \xi_0 \right) \left( \frac{\partial}{\partial \xi_3} + \frac{1}{2} \xi_3 \right) (\psi^0 \xi_0 + \psi^3 \xi_3) \\ &= \frac{iB}{2} (\psi^0 \xi_3 + \psi^3 \xi_0) \\ &= E(\psi^0 \xi_0 + \psi^3 \xi_3), \end{aligned} \quad (6.12)$$

while on Grassmann odd states, we find

$$\begin{aligned} \hat{H}'\psi &= -iB \left( \frac{\partial}{\partial \xi_0} - \frac{1}{2} \xi_0 \right) \left( \frac{\partial}{\partial \xi_3} + \frac{1}{2} \xi_3 \right) (\psi_0 + \psi^{03} \xi_0 \xi_3) \\ &= iB(\psi^{03} + \frac{1}{4} \psi_0 \xi_0 \xi_3) \\ &= E(\psi_0 + \psi^{03} \xi_0 \xi_3). \end{aligned} \quad (6.13)$$

In both cases, the eigenvalues of  $\hat{H}'$  are  $\pm iB/2$ , which means that  $iB$  must be a real number if the Hamiltonian is to be self-adjoint. In turn, this means that if the last term in the Lagrangian (6.1),  $iB\xi_0\xi_3$ , is to be real, then  $\xi_0\xi_3$  must be real, which occurs if one of  $\xi_0$  and  $\xi_3$  is real and one imaginary, since

$$(\xi_0 \xi_3)^* = \xi_3^* \xi_0^* = -\xi_0^* \xi_3^*. \quad (6.14)$$

We will look first at the case  $\xi_3$  real and  $\xi_0$  imaginary, and then examine the difficulties that arise if one, despite the above reasoning, takes both  $\xi_3$  and  $\xi_0$  to be real.

### 1. Imaginary $\xi_0$

If  $\xi_0$  is taken to be imaginary, then, by the constraint  $\pi_0 - \frac{i}{2} \xi_0 \approx 0$ ,  $\pi_0$  will be real. It is easy to check that under the inner product (the measure  $d\xi_0 d\xi_3$  is real)

$$\langle \phi | \psi \rangle = \int \phi^* \psi d\xi_0 d\xi_3 \quad (6.15)$$

with  $\xi_0^* = -\xi_0$  and  $\xi_3^* = \xi_3$  that  $\hat{\xi}_0$  is anti-self-adjoint and  $\hat{\pi}_0$  is self-adjoint,

$$\begin{aligned} \langle \phi | \hat{\xi}_0 \psi \rangle &= -\langle \hat{\xi}_0 \phi | \psi \rangle, \\ \langle \phi | \hat{\pi}_0 \psi \rangle &= \langle \hat{\pi}_0 \phi | \psi \rangle, \end{aligned} \quad (6.16)$$

and similarly that  $\hat{\xi}_3$  and  $\hat{\pi}_3$  are self-adjoint and anti-self-adjoint, respectively. Using this inner product, we find the norms of the definite Grassmann parity states to be

$$\begin{aligned} |(\psi^0 \xi_0 + \psi^3 \xi_3)|^2 &= \psi^{3*} \psi^0 + \psi^{0*} \psi^3, \\ |(\psi_0 + \psi^{03} \xi_0 \xi_3)|^2 &= -\psi^{03*} \psi_0 - \psi_0^* \psi^{03}. \end{aligned} \quad (6.17)$$

The set of physical states is spanned by

$$\begin{aligned} |0\rangle &= \frac{1}{\sqrt{2}}(\xi_0 + \xi_3), \\ |1\rangle &= (1 - \frac{1}{2}\xi_0 \xi_3), \end{aligned} \quad (6.18)$$

while the space of ghost states is spanned by

$$\begin{aligned} |\bar{0}\rangle &= \frac{1}{\sqrt{2}}(\xi_0 - \xi_3), \\ |\bar{1}\rangle &= (1 + \frac{1}{2}\xi_0 \xi_3). \end{aligned} \quad (6.19)$$

All of these states are also eigenstates of the Hamiltonian (6.6). As in the Euclidean cases, the full Schrödinger state space splits into a sum of physical states and ghost states.

It is straightforward to check that the eight Dirac-Gupta-Bleuler conditions are satisfied,

$$\langle \phi_{\text{phys}} | \hat{\varphi}_\mu | \psi_{\text{phys}} \rangle = 0. \quad (6.20)$$

## 2. Real $\xi_0$

If we take the pseudoclassical variables  $\xi_0$  and  $\xi_3$  both to be real, then under our standard inner product,

$$\langle \phi | \psi \rangle = i \int \phi^* \psi d\xi_0 d\xi_3, \quad (6.21)$$

it is straightforward to show that the quantum operators  $\hat{\xi}_\mu$  are self-adjoint and  $\hat{\pi}_\mu$  are anti-self-adjoint,

$$\begin{aligned} \langle \phi | \hat{\xi}_\mu \psi \rangle &= \langle \hat{\xi}_\mu \phi | \psi \rangle, \\ \langle \phi | \hat{\pi}_\mu \psi \rangle &= -\langle \hat{\pi}_\mu \phi | \psi \rangle. \end{aligned} \quad (6.22)$$

The space of positive norm states in this case can be spanned by the (unnormalized) trial basis states

$$\begin{aligned} |0\rangle &= (1 + i\alpha \xi_0 \xi_3), \\ |1\rangle &= (\xi_3 - i\beta \xi_0), \end{aligned} \quad (6.23)$$

with both  $\alpha$  and  $\beta$  fixed positive real numbers, and the orthogonal space of ghost states will then be spanned by trial basis states of the same form but with  $\alpha$  and  $\beta$  replaced by their negatives. If one computes the matrix elements of the constraints to try to find the physical basis, however, one finds

$$\begin{aligned} \langle 1 | \hat{\varphi}_0 | 0 \rangle &= \frac{1}{2} + \alpha\beta = -\langle 0 | \hat{\varphi}_0 | 1 \rangle, \\ \langle 1 | \hat{\varphi}_3 | 0 \rangle &= i \left( \alpha - \frac{\beta}{2} \right) = \langle 0 | \hat{\varphi}_3 | 1 \rangle. \end{aligned} \quad (6.24)$$

The matrix elements between states of the same Grassmann parity automatically vanish.

It is at once apparent that it is impossible to have all of the following conditions hold simultaneously:

1. the inner product is  $\langle \phi | \psi \rangle = i \int \phi^* \psi d\xi_0 d\xi_3$ ;
2. all variables are real, including  $\xi_0^* = \xi_0$ ;
3. all physical states have positive norm; and
4. all matrix elements  $\langle \phi_{\text{phys}} | \hat{\varphi}_\mu | \psi_{\text{phys}} \rangle$  are vanishing.

## B. Reduced phase space quantization

The Poisson brackets amongst the constraints,

$$\{\varphi_\mu, \varphi_\nu\} = -i\eta_{\mu\nu}, \quad (6.25)$$

has inverse matrix  $\Delta^{\mu\nu} = i\eta^{\mu\nu}$ , which leads to the Dirac brackets

$$\{\xi_\mu, \xi_\nu\}_{DB} = -i\eta_{\mu\nu}. \quad (6.26)$$

The reduced phase space is parametrized by the variables  $\xi_0$  and  $\xi_3$ . Again we see that neither of the  $\xi_\mu$  variables can play the part of either position or momentum, but the combinations

$$\begin{aligned} \eta &= \frac{1}{\sqrt{2}}(\xi_3 + \xi_0), \\ \bar{\eta} &= \frac{1}{\sqrt{2}}(\xi_3 - \xi_0), \end{aligned} \quad (6.27)$$

have the correct Dirac brackets to do so,

$$\{\eta, \eta\}_{DB} = \{\bar{\eta}, \bar{\eta}\}_{DB} = 0, \quad \{\eta, \bar{\eta}\}_{DB} = -i. \quad (6.28)$$

States will be functions of just one of the variables, say  $\eta$ ,

$$\psi = \psi(\eta) = \psi_0 + \psi_1 \eta. \quad (6.29)$$

The  $\bar{\eta}$  becomes the momentum operator conjugate to  $\eta$ ,

$$\hat{\bar{\eta}} = \frac{\partial^L}{\partial \eta}. \quad (6.30)$$

The answer to the question of the reality of  $\xi_0$  changes the quantization radically. If  $\xi_0$  is real, then both  $\eta$  and  $\bar{\eta}$  are real, which leads to a quantization similar to the trivial case, whereas if  $\xi_0$  is imaginary, then  $\eta$  and  $\bar{\eta}$  are complex conjugates, which leads to a quantization similar to the reduced phase space quantization in the case of the Euclidean metric.

### 1. Imaginary $\xi_0$

If  $\eta$  and  $\bar{\eta}$  are complex conjugates of each other, the inner product has an integral over both  $\eta$  and  $\bar{\eta}$  as well as a possible measure factor as in (4.41),

$$\langle \phi | \psi \rangle = \int \phi^*(\bar{\eta}) \psi(\eta) \mathcal{M}(\bar{\eta}, \eta) d\eta d\bar{\eta}. \quad (6.31)$$

The operators  $\hat{\eta}$  and  $\hat{\bar{\eta}}$  are adjoints of each other as long as the measure factor  $\mathcal{M}$  is again  $e^{\bar{\eta}\eta}$ . The Hamiltonian  $H = iB\xi_0\xi_3 = iB(\eta\bar{\eta} - \bar{\eta}\eta)/2$  becomes the operator

$$\hat{H} = \frac{iB}{2} \left( \eta \frac{\partial^L}{\partial \eta} - \frac{\partial^L}{\partial \eta} \eta \right), \quad (6.32)$$

whose eigenstates and eigenvalues are given by

$$\begin{aligned} |0\rangle &= 1, \\ |1\rangle &= \eta, \\ \hat{H}|0\rangle &= -\frac{iB}{2}|0\rangle, \\ \hat{H}|1\rangle &= \frac{iB}{2}|1\rangle. \end{aligned} \quad (6.33)$$

### 2. Real $\xi_0$

If  $\eta$  and  $\bar{\eta}$  are real, then one, say  $\eta$ , can be taken to be a position coordinate and the other,  $\bar{\eta}$ , its momentum. States are then functions of  $\eta$  alone,

$$\psi = \psi(\eta) = \psi_0 + \psi_1 \eta, \quad (6.34)$$

and the inner product is the same as in section III. Unlike in section III, there are no constraints to impose, because they were eliminated classically before quantization, but physical states must still have positive norm,

$$|\psi|^2 = \int \psi^*(\eta) \psi(\eta) d\eta = 2 \operatorname{Re}(\psi_0^* \psi_1) > 0, \quad (6.35)$$

which restricts the states to be proportional to

$$\psi_{\text{phys}} = 1 + \psi_1 \eta, \quad (6.36)$$

with  $\psi_1$  having a positive real part. The self-adjointness of  $\hat{\eta}$  is clear, and one can easily check that  $\hat{\bar{\eta}}$  is self-adjoint.

So it would appear that when we take  $\xi_0^* = \xi_0$  in reduced phase space quantization, we are allowed the full machinery of the Schrödinger representation, and the Hamiltonian (6.32) is the same as in the case  $\xi_0^* = -\xi_0$ , but in this case the Hamiltonian has no normalizable eigenstates and the space of states of form (6.36) is not a Hilbert space. The states (6.36) can approach eigenstates as  $\psi_1 \rightarrow \infty$  or  $\psi_1 \rightarrow 0$ , but the limit states are not normalizable.

## VII. FOUR ANTICOMMUTING VARIABLES WITH LORENTZIAN METRIC

As a final example, we consider a Lorentzian action that is well behaved,

$$L = \frac{i}{2} \xi_\mu \dot{\xi}^\mu - i F^{\mu\nu} \xi_\mu \xi_\nu, \quad (7.1)$$

with  $F^{\mu\nu}$  consisting purely of a magnetic field (i.e.,  $F_{0i} = F_{i0} = 0$ ), which we take to be in the third direction. We again quantize by finding the constraints, the Hamiltonian, and then, depending on the method, the set of position coordinates, inner product and the set of (physical) states.

As usual, the constraints are given by Eq. (6.3), but this time the index  $\mu$  can run from 0 to 3. The Hamiltonian is again the same as (4.30), but the wave functions can depend on all four  $\xi_\mu$  variables. Because this model can be viewed as the sum of the two-variable Euclidean case (with  $\xi_1$  and  $\xi_2$ ) and the two-variable Lorentzian case (with  $\xi_0$  and  $\xi_3$ ), we can combine previous results to get the states and energies. Since, unlike the  $\xi_3$  kinetic term, the sign of the  $\xi_0$  kinetic term is opposite the sign in Eq. (3.1), then, as discussed in Sections III and VI, the Dirac-Gupta-Bleuler quantization will only work with  $\xi_0$  imaginary. The basis of states for the full theory can be obtained by taking products of basis states of the Euclidean two-variable  $\xi_1$  and  $\xi_2$  system and the basis states of the Lorentzian two-variable  $\xi_3$  and  $\xi_0$  system, respectively. Products of two constituent physical states or two constituent ghost states will give physical states in the final model, while a product of one ghost state and one physical state will give a ghost state in the final model.

### A. Dirac-Gupta-Bleuler quantization

The inner product on the space of states is

$$\langle \phi | \psi \rangle = i \int \phi^* \psi d\xi_0 d\xi_1 d\xi_2 d\xi_3. \quad (7.2)$$

The positive norm physical states are

$$\begin{aligned} |0\rangle &= (1 + \frac{i}{2} \xi_1 \xi_2)(1 - \frac{1}{2} \xi_0 \xi_3), \\ |1\rangle &= \frac{1}{\sqrt{2}} (1 + \frac{i}{2} \xi_1 \xi_2)(\xi_0 + \xi_3), \\ |2\rangle &= \frac{1}{\sqrt{2}} (\xi_1 + i \xi_2)(1 - \frac{1}{2} \xi_0 \xi_3), \\ |3\rangle &= \frac{1}{2} (\xi_1 + i \xi_2)(\xi_0 + \xi_3), \\ |0'\rangle &= (1 - \frac{i}{2} \xi_1 \xi_2)(1 + \frac{1}{2} \xi_0 \xi_3), \\ |1'\rangle &= \frac{1}{\sqrt{2}} (1 - \frac{i}{2} \xi_1 \xi_2)(\xi_0 - \xi_3), \end{aligned}$$

$$\begin{aligned}
|2'\rangle &= \frac{1}{\sqrt{2}}(\xi_1 - i\xi_2)(1 + \frac{1}{2}\xi_0\xi_3), \\
|3'\rangle &= \frac{1}{2}(\xi_1 - i\xi_2)(\xi_0 - \xi_3),
\end{aligned} \tag{7.3}$$

while the ghost states are

$$\begin{aligned}
|\bar{0}\rangle &= (1 + \frac{i}{2}\xi_1\xi_2)(1 + \frac{1}{2}\xi_0\xi_3), \\
|\bar{1}\rangle &= \frac{1}{\sqrt{2}}(1 - \frac{i}{2}\xi_1\xi_2)(\xi_0 + \xi_3), \\
|\bar{2}\rangle &= \frac{1}{\sqrt{2}}(\xi_1 - i\xi_2)(1 - \frac{1}{2}\xi_0\xi_3), \\
|\bar{3}\rangle &= \frac{1}{2}(\xi_1 - i\xi_2)(\xi_0 + \xi_3), \\
|\bar{0}'\rangle &= (1 - \frac{i}{2}\xi_1\xi_2)(1 - \frac{1}{2}\xi_0\xi_3), \\
|\bar{1}'\rangle &= \frac{1}{\sqrt{2}}(1 + \frac{i}{2}\xi_1\xi_2)(\xi_0 - \xi_3), \\
|\bar{2}'\rangle &= \frac{1}{\sqrt{2}}(\xi_1 + i\xi_2)(1 + \frac{1}{2}\xi_0\xi_3), \\
|\bar{3}'\rangle &= \frac{1}{2}(\xi_1 + i\xi_2)(\xi_0 - \xi_3).
\end{aligned} \tag{7.4}$$

The energies of these states are

$$\begin{aligned}
\hat{H}'|0\rangle &= -\frac{B}{2}|0\rangle, & \hat{H}'|0'\rangle &= -\frac{B}{2}|0'\rangle, \\
\hat{H}'|1\rangle &= -\frac{B}{2}|1\rangle, & \hat{H}'|1'\rangle &= -\frac{B}{2}|1'\rangle, \\
\hat{H}'|2\rangle &= +\frac{B}{2}|2\rangle, & \hat{H}'|2'\rangle &= +\frac{B}{2}|2'\rangle, \\
\hat{H}'|3\rangle &= +\frac{B}{2}|3\rangle, & \hat{H}'|3'\rangle &= +\frac{B}{2}|3'\rangle.
\end{aligned} \tag{7.5}$$

### B. Reduced phase space quantization

No new work need be done to construct the Dirac bracket. We find that the four  $\xi_\mu$  will cover the whole reduced phase space and their Dirac brackets are

$$\{\xi_\mu, \xi_\nu\} = -i\eta_{\mu\nu}. \tag{7.6}$$

In a similar manner to the previous cases, we can take position coordinates to be

$$\begin{aligned}
\eta_1 &= \frac{1}{\sqrt{2}}(\xi_3 + \xi_0), \\
\eta_2 &= \frac{1}{\sqrt{2}}(\xi_1 + i\xi_2),
\end{aligned} \tag{7.7}$$

and their momenta to be

$$\begin{aligned}
\bar{\eta}_1 &= \frac{1}{\sqrt{2}}(\xi_3 - \xi_0), \\
\bar{\eta}_2 &= \frac{1}{\sqrt{2}}(\xi_1 - i\xi_2).
\end{aligned} \tag{7.8}$$

The general state,

$$\psi = \psi_0 + \psi_1\eta_1 + \psi_2\eta_2 + \psi_3\eta_1\eta_2, \tag{7.9}$$

has four complex coefficients,  $\psi_a$ . The inner product on the space of states will depend on the reality properties of  $\xi_0$ , and hence the reality properties of  $\eta_1$ . If  $\xi_0$  is taken to be imaginary as in section VI, then the inner product will be

$$\begin{aligned}
\langle\phi|\psi\rangle &= \int \phi^*(\bar{\eta}_1, \bar{\eta}_2)\psi(\eta_1, \eta_2) \mathcal{M} d\eta_1 d\bar{\eta}_1 d\eta_2 d\bar{\eta}_2 \\
&= \phi_a^* \psi_a,
\end{aligned} \tag{7.10}$$

with the measure factor  $\mathcal{M}(\eta_1, \bar{\eta}_1, \eta_2, \bar{\eta}_2) = \exp(\bar{\eta}_1\eta_1 + \bar{\eta}_2\eta_2)$  necessary to give the quantum operators the correct self-adjointness properties that follow from the reality properties of their classical counterparts, just as in the case of two variables.

## VIII. DISCUSSION AND CONCLUSIONS

The Schrödinger realization of the quantized pseudoclassical theories has been little studied in comparison to the path integral quantization. The existence of the Schrödinger realization has been assumed by Bordi, Casalbuoni, and Barducci [18, 19], who also first found the physical states given in Eq. (3.8). Delbourgo [27] considered nonrelativistic spin systems represented by the Schrödinger picture quantum mechanics of two anticommuting variables, and relativistic systems represented by four anticommuting variables. He also considered more general involutions on these variables.

The physical states (7.3) have been looked at from an abstract point of view by Mankoč Borštnik [28, 29] and Mankoč Borštnik and Nielsen [30–32], which is closer to a Dirac-Gupta-Bleuler quantization rather than to the reduced phase space quantization that the abstract approach of Berezin and Marinov [7] closely resembles.

Quantized pseudoclassical systems in the Schrödinger realization using the Dirac-Gupta-Bleuler method have a detailed interdependence of the reality of the variables, the Grassmann parity of the wave functions, and the split between physical and ghost states.

We have seen that in a Dirac-Gupta-Bleuler quantization, adding one more real Grassmann coordinate to a system with an even number of Grassmann variables has two effects. The first is that the number of physical states will double because the ghost state for the new variable can pair with ghost states of the previous system to make physical states in the combined system. In terms of the quantum mechanics, these new states are in a different superselection sector and may be ignored. The second effect is to make the physical states be of mixed Grassmann parity, because the “measure” in the integral will now have odd Grassmann parity. By contrast, the reduced phase space quantization has a positive definite inner product and so always produces an irreducible

representation of the Clifford algebra; adding one more Grassmann coordinate to the system does not lead to a doubling of the number of physical states.

We have also seen that the behavior of the Grassmann coordinates under the involution, in other words, whether the variables are real or imaginary, has an effect on the quantum system. In the trivial case, the quantum mechanics of an imaginary Grassmann variable cannot have a Schrödinger realization unless the kinetic term is negative because the constraint otherwise cannot be imposed. In a two-variable system with a Lorentzian kinetic energy both a Dirac-Gupta-Bleuler and a reduced phase space quantization can be done if one variable is taken to be real and one imaginary. If both are taken to be real, then neither a Dirac-Gupta-Bleuler quantization in the Schrödinger realization nor a reduced phase space quantization exists; in the first there are no physical states, while in the second the space of normalizable states fails to be a Hilbert space. As the behavior of the variables under the involution in the pseudoclassical theory determines the (anti-)self-adjointness properties of the corresponding quantum operators, the timelike  $\xi_0$  should have reality properties opposite to the spacelike  $\xi_i$  because their corresponding quantum operators, the gamma matrices  $\gamma^0$  and  $\gamma^i$ , have opposite Hermiticity properties. As Berezin and Marinov argue, the  $\xi_0$  needs to be present for manifest Lorentz invariance, but the pseudoclassical Dirac equation is motivated by finding a way to remove the  $\xi_0$  from the system in a covariant way.

## Appendix A: Gamma representation

In the 3 + 1 dimensional case, the Dirac primed variables

$$\xi^{\mu'} = \xi^\mu - \{\xi^\mu, \varphi_\alpha\} \Delta^{\alpha\beta} \varphi_\beta = \xi^\mu - i(\pi^\mu - \frac{i}{2} \xi^\mu),$$

$$\hat{\xi}^{\mu'} = \frac{\partial^L}{\partial \xi_\mu} + \frac{1}{2} \eta^{\mu\nu} \xi_\nu, \quad (A1)$$

satisfy the anticommutation relations

$$\hat{\xi}^{\mu'} \hat{\xi}^{\nu'} + \hat{\xi}^{\nu'} \hat{\xi}^{\mu'} = \eta^{\mu\nu} = \text{diag}(-, +, +, +). \quad (A2)$$

After scaling, the  $\xi^{\mu'}$  can be represented by the Dirac gamma matrices;  $\sqrt{2} \hat{\xi}^{\mu'} \rightarrow \gamma^\mu$ . In the unprimed physical basis (7.3), we define

$$\psi_0|0\rangle + \psi_1|1\rangle + \psi_2|2\rangle + \psi_3|3\rangle = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \quad (A3)$$

and the representation of the  $\hat{\xi}^{\mu'}$  are

$$\begin{aligned} \sqrt{2} \hat{\xi}^{0'} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = -i\sigma_3 \otimes \sigma_2, \\ \sqrt{2} \hat{\xi}^{1'} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \sigma_1 \otimes \mathbf{1}, \\ \sqrt{2} \hat{\xi}^{2'} &= \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} = \sigma_2 \otimes \mathbf{1}, \\ \sqrt{2} \hat{\xi}^{3'} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \sigma_3 \otimes \sigma_1. \end{aligned} \quad (A4)$$

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